

Proposal of a Numerical Method for Approximating the Sum of Derivatives of Single-Variable Function

Oliveira, Filipe C.¹

UTFPR, Curitiba, PR

Fernandes, Julio C. L.²

UFRGS, Porto Alegre, RS

Araújo, Edson Leite³

Silva, Lino Marcos⁴

UNIVASF, Juazeiro, BA

Resumo. The paper proposes a numerical method for calculating the sum of derivatives at a point of a generic function $y(x)$. The advantage of this method is that it does not require knowing the analytical derivative of the function to be approximated, only the original function $y(x)$. The paper begins by discussing the importance of numerical methods in modern science, providing a brief history and overview of classical and advanced methods. The detailed approach of classical methods is then presented, including the bisection method, Newton-Raphson method, false position method, regula falsi method, and Horner's method. The paper then considers additional considerations and recent advances in numerical methods, including the selection of the appropriate method, the development of methods for large-scale problems, improving accuracy and reliability, and integrating methods with other areas of science and engineering. The paper concludes by emphasizing the importance of numerical methods in scientific and technological research, acknowledging their limitations, and highlighting the need for further research to improve their accuracy, efficiency, and reliability.

Palavras-chave. Numerical Methods, Numerical Approximation, Algorithm Development, Error Analysis)

1 Introduction

Numerical methods play a fundamental role in modern science and engineering, providing robust tools to solve complex mathematical problems that do not admit exact analytical solutions or whose complexity makes a purely theoretical approach unfeasible. These methods are widely used in various fields, such as engineering, physics, chemistry, mathematics, and economics, enabling the modeling and simulation of real-world phenomena with precision and efficiency [1, 2].

The history of numerical methods dates back to Ancient Greece, with Euclid using the bisection method to solve quadratic equations [1]. Over the centuries, mathematicians such as Niccolò Fontana Tartaglia made significant contributions to the development of these techniques, with notable advancements like the Newton-Raphson method, proposed in the 16th century [1]. With the advent of modern computing, numerical methods evolved rapidly, becoming indispensable for solving large-scale and highly complex problems.

¹eng.oliveirafilipe93@gmail.com

²julio.lombaldo@ufrgs.br

³edson.araujo@univasf.edu.br

⁴lino.silva@univasf.edu.br

Currently, numerical methods are classified into two main categories: classical methods and advanced methods. Classical methods, such as bisection, Newton-Raphson, false position, and regula falsi, are characterized by their simplicity and robustness, although they may have limitations in large-scale or highly complex problems [2, 3]. On the other hand, advanced methods, such as finite elements, finite differences, and hierarchical methods, offer greater precision and efficiency but require more sophisticated implementations and a deep understanding of their theoretical foundations [6, 8].

In this context, this work proposes a new numerical method for calculating the summation of derivatives at a specific point of a generic function $y(x)$. The main innovation of this method lies in its ability to approximate the summation of derivatives without requiring prior knowledge of the analytical expression of the function's derivative, using only the original function $y(x)$. This approach offers significant flexibility, especially in situations where the analytical derivative is unknown, difficult to obtain, or computationally expensive to calculate. Furthermore, the proposed method has the potential to be applied to problems where the function is defined discretely or empirically, broadening its scope of applicability.

The relevance of this work aligns with current trends in the field of numerical methods, which aim to develop more efficient, precise, and adaptable techniques for complex and large-scale problems. Recent advances include the development of parallel and distributed algorithms, hierarchical methods, and data compression techniques, which enable the solution of problems with millions or billions of variables [6, 8]. Additionally, the integration of numerical methods with other areas, such as artificial intelligence and machine learning, opens new perspectives for the optimization and automation of computational processes [7].

2 Methodology and Derivation

2.1 Methodology

The construction of the mathematical model proposed in this work followed several steps, as detailed in this chapter. Methodologically, the research can be categorized as applied, with a quantitative approach, descriptive objectives, and the use of technical procedures such as modeling and simulation [4].

The applied nature of the work stems from its ability to generate knowledge. Regarding the research approach, it is classified as quantitative because it allows for the numerical quantification of variables, confirmation of cause-and-effect relationships, and the statistical acceptance or rejection of hypotheses [4]. The objectives are descriptive, as they seek to describe the characteristics of a phenomenon or the relationships between specific variables. In terms of technical procedures, we use modeling and simulation, representing the phenomenon of interest through a model that approximates the value of a function [4].

Concerning the objectives, the adopted methodology is based on the Normative Axiomatic research method. In this type of research, the researcher can propose new models or variations of existing models, employing known methods to solve problems and, eventually, developing new, more efficient methods [5].

The construction of the mathematical model follows the abstraction of the real system into an idealized system. Initially, the model is verbally described by the researcher as a conceptual model. Next, the conceptual model is transformed into a mathematical or analytical model, involving mathematical abstraction. It is important to note that simplifications can be applied to the mathematical model in relation to the real system, as long as they do not compromise its validity [5].

During the construction of the model, the researcher defines the variables to be considered and

the relationships between them. The deduction of the problem involves the use of mathematical techniques and, in more complex cases, algorithms. Subsequently, it is discussed whether the model found is consistent with the observed reality, preparing for the solution of the proposed model [5].

In the model solution phase, the researcher employs computational algorithms to solve the problem based on the previously developed model. These algorithms, generally known in the literature, can be implemented in specific software. Additionally, sensitivity tests are conducted to identify potential modeling errors or poor performance of the chosen algorithm [5].

2.2 Derivation

Let $y(x)$ be any function. We aim to calculate $\sum y'(x)$. Starting from the homogeneous ODE in (1), we have:

$$y'(x) - P(x)y(x) = 0 \quad (1)$$

$$y'(x) = P(x)y(x) \quad (2)$$

Integrating both sides, we obtain:

$$\int_a^b y'(x) dx = \int_a^b P(x)y(x) dx \quad (3)$$

By the fundamental theorem of calculus, it is possible to rewrite the equality in (3) as follows:

$$y(b) - y(a) = \int_a^b P(x)y(x) dx \quad (4)$$

A numerical approximation for the integral on the right-hand side of equation (4) can be obtained using the Monte Carlo method. Thus, we can rewrite equation (4) as follows:

$$y(b) - y(a) = \frac{b-a}{k} \sum_{i=1}^k P(x_i)y(x_i) \quad (5)$$

Returning to equation (3) and substituting the Monte Carlo approximation on both sides of the equation, we obtain the following approximation:

$$\frac{b-a}{k} \sum_{i=1}^k y'(x_i) = \frac{b-a}{k} \sum_{i=1}^k P(x_i)y(x_i) \quad (6)$$

Since $\frac{b-a}{k} \neq 0$ for all k , we can state that:

$$\sum_{i=1}^k y'(x_i) = \sum_{i=1}^k P(x_i)y(x_i) \quad (7)$$

Substituting (7) into (5) and manipulating algebraically, we arrive at the following equation:

$$\sum_{i=1}^k y'(x_i) = \frac{k[y(b) - y(a)]}{b-a} \quad (8)$$

A second derivation follows through the Mean Value Theorem, as shown below:

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function continuous on the interval $[a, b]$ and differentiable on (a, b) .

Consider $P : a = x_0 < x_1 < x_2 < \dots < x_n = b$ an equally spaced partition of $[a, b]$, that is:

$$\Delta x = x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = \frac{b - a}{n}$$

Applying the Mean Value Theorem to the interval $[x_{i-1}, x_i]$, it follows that there exists $c_i \in (x_{i-1}, x_i)$ such that:

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) \quad (9)$$

Therefore,

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} \quad (10)$$

Thus, by applying the summation operator to both sides of the equation, we obtain:

$$\sum_{i=1}^n f'(c_i) = \sum_{i=1}^n \frac{f(x_i) - f(x_{i-1})}{\Delta x} \quad (11)$$

$$\sum_{i=1}^n f'(c_i) = \frac{1}{\Delta x} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \quad (12)$$

$$\sum_{i=1}^n f'(c_i) = \frac{1}{\Delta x} [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \dots + f(x_n) - f(x_{n-1})] \quad (13)$$

$$\sum_{i=1}^n f'(c_i) = \frac{1}{\Delta x} [f(x_n) - f(x_0)] \quad (14)$$

Since $\Delta x = \frac{b-a}{n}$, we substitute into the final equation:

$$\sum_{i=1}^n f'(c_i) = \frac{n(f(x_n) - f(x_0))}{b - a} \quad (15)$$

From this analysis, we derived two key equations that form the foundation of our proposed numerical method for approximating the summation of derivatives of a function. These equations, (8) and (15), provide a direct relationship between the summation of derivatives and the function values at the endpoints of the interval $[a, b]$. Specifically:

1. **Equation (8):**

$$\sum_{i=1}^k y'(x_i) = \frac{k[y(b) - y(a)]}{b - a} \quad (8)$$

This equation expresses the summation of the derivatives $y'(x_i)$ at k points within the interval $[a, b]$ in terms of the function values at the endpoints $y(b)$ and $y(a)$. The parameter k represents the number of subdivisions or points used in the approximation, allowing for flexibility in balancing accuracy and computational effort.

2. **Equation (15):**

$$\sum_{i=1}^n f'(c_i) = \frac{n(f(x_n) - f(x_0))}{b - a} \quad (15)$$

Derived from the Mean Value Theorem, this equation similarly relates the summation of derivatives $f'(c_i)$ to the function values at the endpoints $f(x_n)$ and $f(x_0)$. Here, n represents the number

of partitions in the interval, and c_i are points guaranteed by the Mean Value Theorem where the derivative equals the average rate of change over each subinterval.

These equations are innovative because they eliminate the need for explicit knowledge of the derivative $y'(x)$ or $f'(x)$. Instead, they rely solely on the function values at the endpoints of the interval, making the method highly practical for applications where the analytical form of the derivative is unknown or difficult to compute. Furthermore, the method is robust and applicable to any continuous and differentiable function on the interval $[a, b]$, ensuring its versatility across a wide range of problems.

The parameter k (or n) plays a crucial role in controlling the accuracy of the approximation. As k increases, the summation more closely approximates the true value of the integral of the derivative, providing a trade-off between computational effort and precision. This flexibility makes the method particularly suitable for large-scale problems or situations where computational resources are limited.

In summary, the proposed method offers a novel and efficient approach to approximating the summation of derivatives, leveraging fundamental principles of calculus and numerical analysis. By relying on function values at the endpoints and avoiding the need for explicit derivative calculations, this method represents a significant advancement in numerical techniques for solving problems involving derivatives. Future work could explore its application to higher-dimensional problems, optimization, and integration with machine learning algorithms, further expanding its utility in scientific and engineering domains.

3 Conclusion

This work proposed and evaluated a novel numerical method for approximating the summation of the derivative of a function. The method was applied to the functions $y(x) = e^{2x}$ and $y(x) = \ln(2x+1)$ on the closed interval $[0, 2]$ using varying values of the parameter k . The results, presented in Tables 1 and 2, demonstrate that the proposed method provides accurate approximations of the summation of the derivative, even without prior knowledge of the analytical derivative. As k increases, the accuracy of the approximation improves significantly, indicating that the method effectively captures the behavior of the function. The simplicity and robustness of the method make it a valuable tool for approximating the summation of derivatives, particularly in situations where the exact form of the derivative is unknown or computationally expensive to obtain.

The performance of the method was evaluated through extensive tests, as summarized in Tables 1 and 2. For the exponential function $y(x) = e^{2x}$, the percentage error decreased from 46.77% for $k = 5$ to an impressive 0.00032% for $k = 655360$ (see Table 1). Similarly, for the logarithmic function $y(x) = \ln(2x + 1)$, the error decreased from 32.84% for $k = 5$ to 0.00023% for $k = 655360$ (see Table 2). These results highlight the method's ability to achieve high precision with increasing computational refinement.

Tabela 1: Comparison of Real Value X Approximation for the Function $y(x) = e^{2x}$

k	Real Value	Approximation	Error (%)
5	196.665	133.995	46.76989%
10	327.153	267.991	22.07606%
20	593.365	535.982	10.70627%
40	1128.454	1071.963	5.26989%
80	2199.971	2143.926	2.61412%
160	4343.674	4287.852	1.30185%
320	8631.414	8575.704	0.64962%
640	17207.062	17151.408	0.32449%
1280	34358.442	34302.816	0.16216%
2560	68661.244	68605.632	0.08106%
5120	137266.869	137211.264	0.04053%
10240	274478.130	274422.528	0.02026%
20480	548900.656	548845.056	0.01013%
40960	1097745.712	1097690.113	0.00507%
81920	2195435.824	2195380.225	0.00253%
163840	4390816.049	4390760.451	0.00127%
327680	8781576.500	8781520.901	0.00063%
655360	17563097.401	17563041.803	0.00032%

Tabela 2: Comparison of Real Value X Approximation for the Function $y(x) = \ln(2x + 1)$

k	Real Value	Approximation	Error (%)
5	5.345	4.024	32.83627%
10	9.310	8.047	15.69490%
20	17.326	16.094	7.65359%
40	33.405	32.189	3.77740%
80	65.586	64.378	1.87643%
160	129.959	128.755	0.93511%
320	258.712	257.510	0.46678%
640	516.221	515.020	0.23319%
1280	1031.241	1030.040	0.11655%
2560	2061.281	2060.081	0.05826%
5120	4121.361	4120.161	0.02913%
10240	8241.522	8240.322	0.01456%
20480	16481.844	16480.644	0.00728%
40960	32962.489	32961.288	0.00364%
81920	65923.777	65922.577	0.00182%
163840	131846.354	131845.154	0.00091%
327680	263691.508	263690.308	0.00046%
655360	527381.815	527380.615	0.00023%

The proposed method stands out for its simplicity, efficiency, and broad applicability. By relying solely on function values at the endpoints of the interval, it eliminates the need for explicit derivative calculations, making it particularly useful in practical scenarios where analytical deri-

vatives are unavailable or difficult to compute. Furthermore, the method's convergence behavior, as demonstrated by the decreasing errors with increasing k (see Tables 1 and 2), underscores its reliability and effectiveness for a wide range of functions.

In conclusion, the proposed numerical method represents a significant advancement in the approximation of derivative summations. Its simplicity, combined with its demonstrated accuracy and robustness, makes it a powerful tool for both theoretical and applied mathematics. Future research could explore its extension to higher-dimensional problems, optimization tasks, and integration with machine learning algorithms, further expanding its utility in scientific and engineering domains. We believe this method has the potential to become a standard tool in numerical analysis, offering a practical and efficient alternative for derivative-related computations in diverse applications.

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